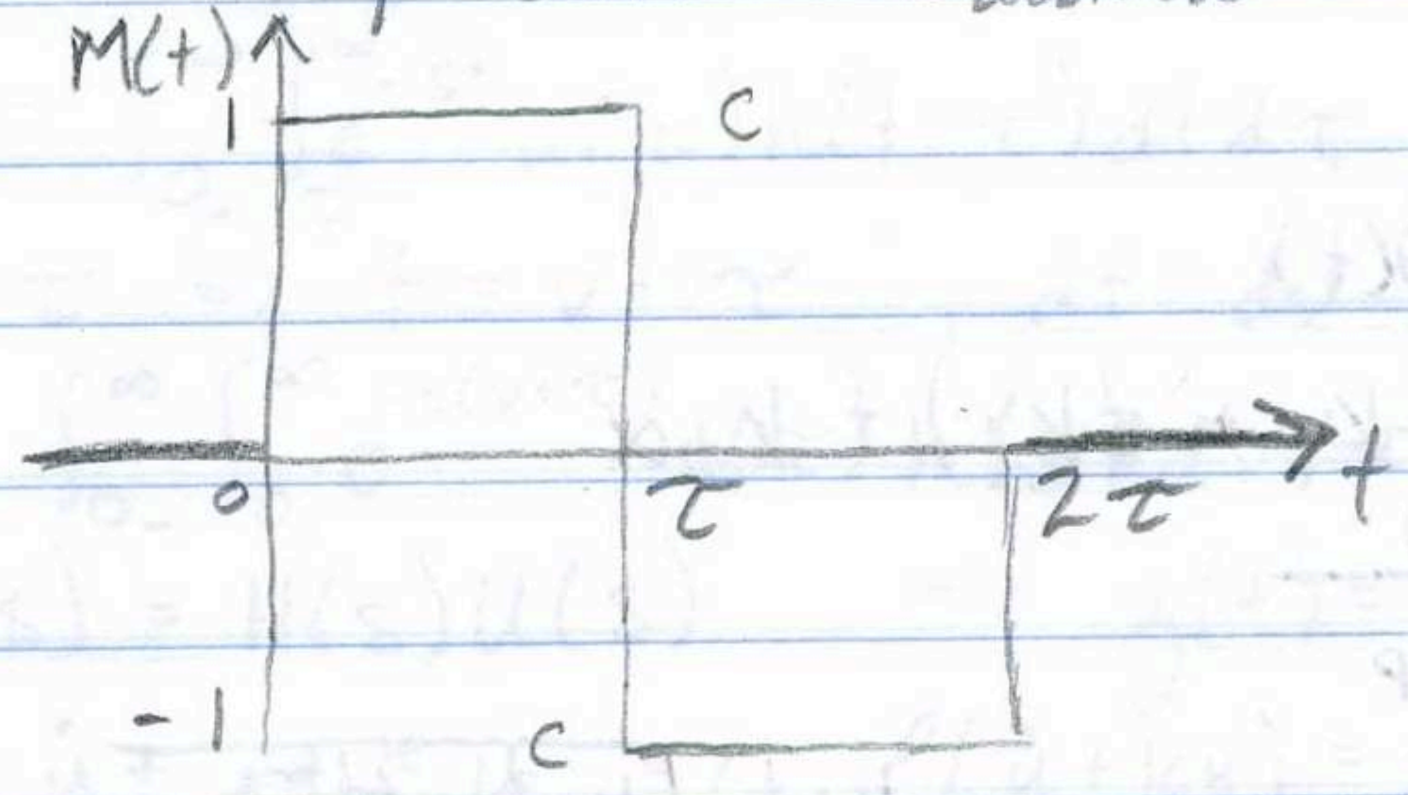
3-27-08

Single-Axis Attitude Control Example

(1) $J\ddot{\theta} = M$ where J = mass moment of inertia ; M = applied torque

$\theta(0) = \alpha$ goal: $\theta = 0$

$\dot{\theta}(0) = 0$, error $= \theta_{desired} - \theta(0) = -\alpha$



$$M(t) = c\left(u_s(t) - 2u_s(t-\tau) + u_s(t-2\tau)\right)$$

$$\mathcal{L}(f(t-\tau)) = e^{-s\tau}F(s)$$

$$\Rightarrow u_s(s) = \frac{1}{s}$$

$$\Rightarrow M(s) = c\left(\frac{1}{s} - 2e^{-s\tau}\cdot\frac{1}{s} + e^{-2s\tau}\frac{1}{s}\right)$$

$$= \frac{c}{s}\left(1 - 2e^{-s\tau} + e^{-2s\tau}\right)$$

$$M(s) = \mathcal{L}(J\ddot{\theta}) = J(s^2\theta(s) - s\theta(0) - \cancel{\dot{\theta}(0)}) = Js^2\theta(s) - Js\alpha$$

$$\Rightarrow \theta(s) = \frac{\alpha}{s} + \frac{c}{Js^3}\left(1 - 2e^{-s\tau} + e^{-2s\tau}\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^3}\right) = u_s(t)\left(\frac{t^2}{2}\right)$$

$$\Rightarrow \theta(t) = \alpha u_s(t) + \frac{c}{J}\left(\frac{t^2}{2}u_s(t) - 2\frac{(t-\tau)^2}{2}u_s(t-\tau) + \frac{(t-2\tau)^2}{2}u_s(t-2\tau)\right)$$

For $t > 2\tau$:

$$\theta(t) = \alpha + \frac{c\tau^2}{J} \qquad 2 = 2\tau \Rightarrow \tau = 1$$

Problems with open-loop control

1. $\theta(0) = \alpha + \Delta\alpha$

$$(\alpha + \Delta\alpha) - \frac{\alpha J(1)^2}{J} = \Delta\alpha \neq 0 \qquad *\text{error terms}$$

2. $J = J + \Delta J$

$$\alpha - \frac{\alpha J(1)^2}{J + \Delta J} = \alpha\left(\frac{\Delta J}{J + \Delta J}\right) \neq 0$$

3. Disturbances : $M(t) = M_c(t) + M_d(t) = \delta(t)$

$$\theta_d(t) = \mathcal{L}^{-1}\left(\frac{1}{Js^2}\right) = \frac{t}{J} \Rightarrow \theta(t) = \underline{\qquad} +$$

*error increases with increase in time

Closed-Loop Control

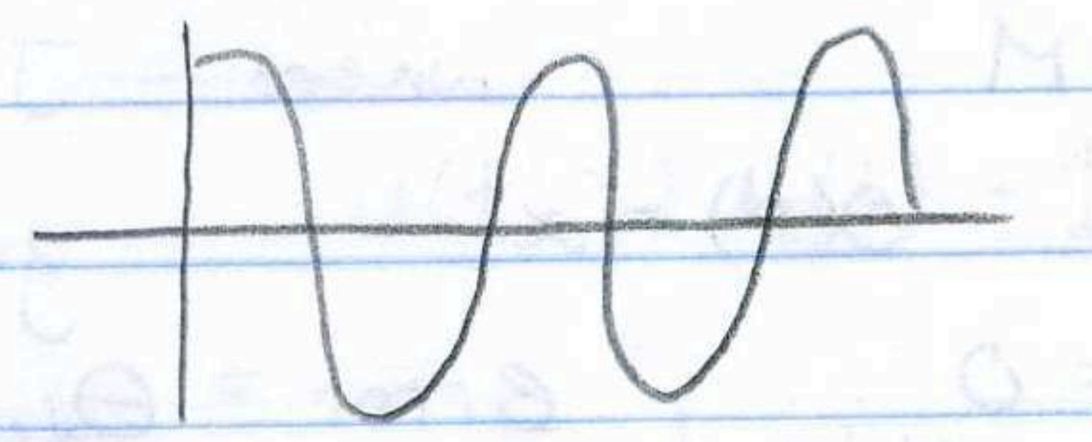
(i) proportional feedback

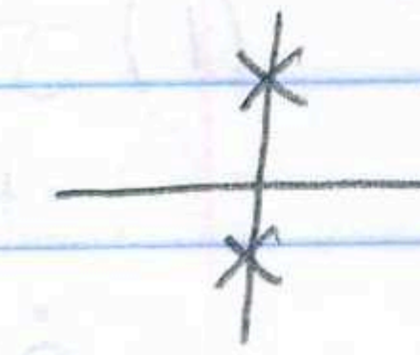
$$M_c(t) = -k_p\theta(t)$$

$$Js^2\Theta(s) - Js\alpha = -k_p\Theta(s) \Rightarrow \Theta(s) = \frac{\alpha J s}{Js^2 + k_p} = \frac{\alpha s}{s^2 + k_p/J}$$

$\omega = \sqrt{k_p/J}$

$\Rightarrow \Theta(t) = \alpha \cos \omega t$





marginally stable

(ii) PD controller

$$M_c(t) = -k_p\Theta(t) - k_d\dot{\Theta}(t)$$

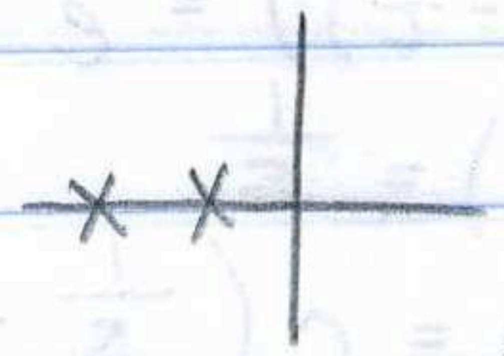
$$Js^2\Theta(s) - Js\alpha = -\Theta(s)(k_p + sk_d) + k_d\alpha$$

$$\Rightarrow \Theta(s) = \frac{\alpha(Js + k_d)}{Js^2 + k_d s + k_p}$$

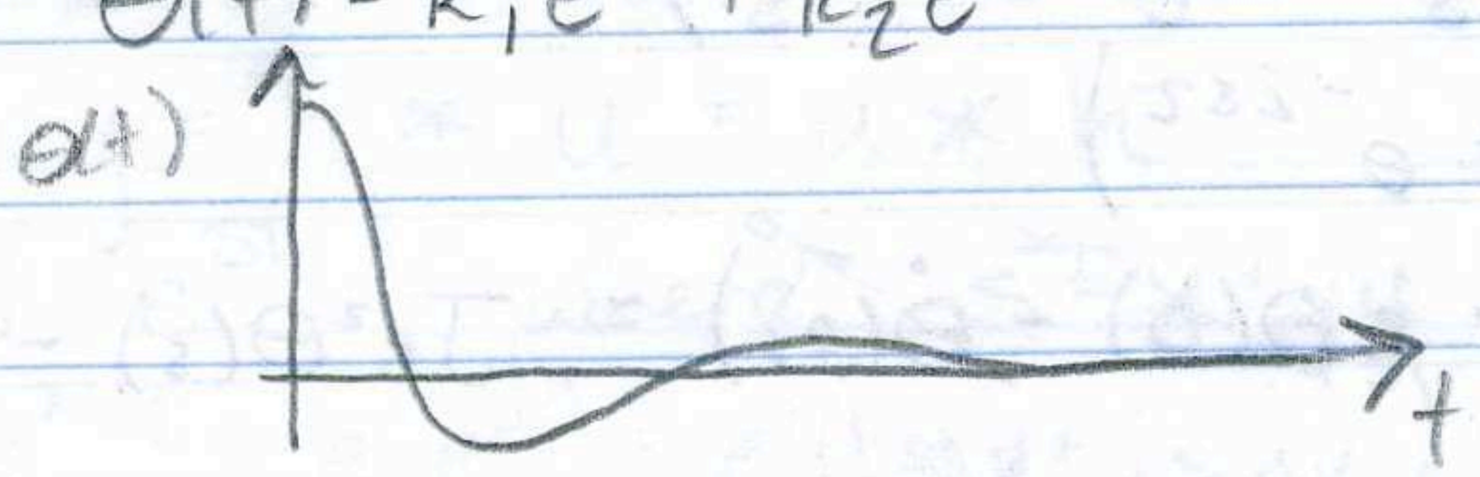
$s^2 + 2\xi\omega_n s + \omega_n^2$, $\quad s_{1,2} = \dfrac{-k_d \pm \sqrt{k_d^2 - 4Jk_p}}{2J}$

$s = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$

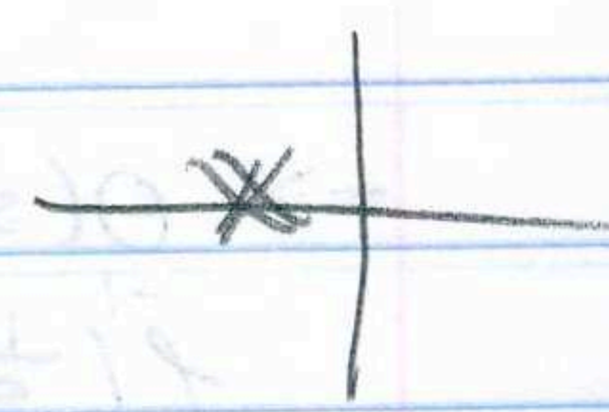
overdamped: $\xi > 1 \Rightarrow k_d^2 > 4Jk_p$



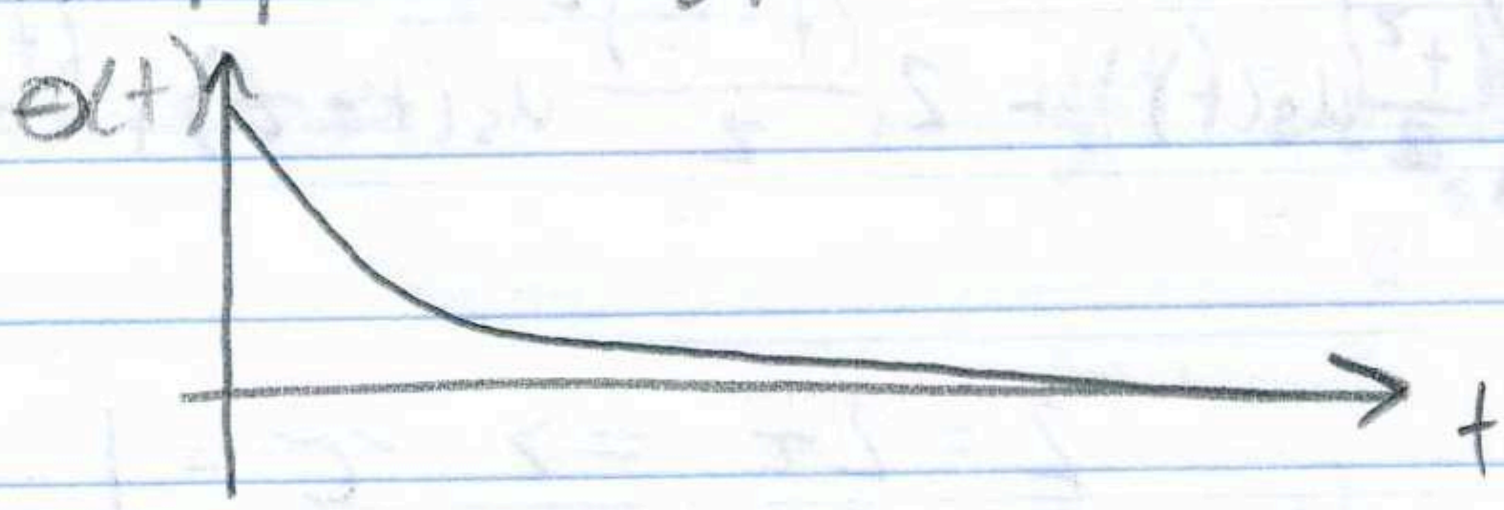
$\Theta(t) = k_1 e^{s_1 t} + k_2 e^{s_2 t}$



critically damped: $\xi = 1$, $k_d^2 = 4Jk_p \Rightarrow s_1 = s_2$

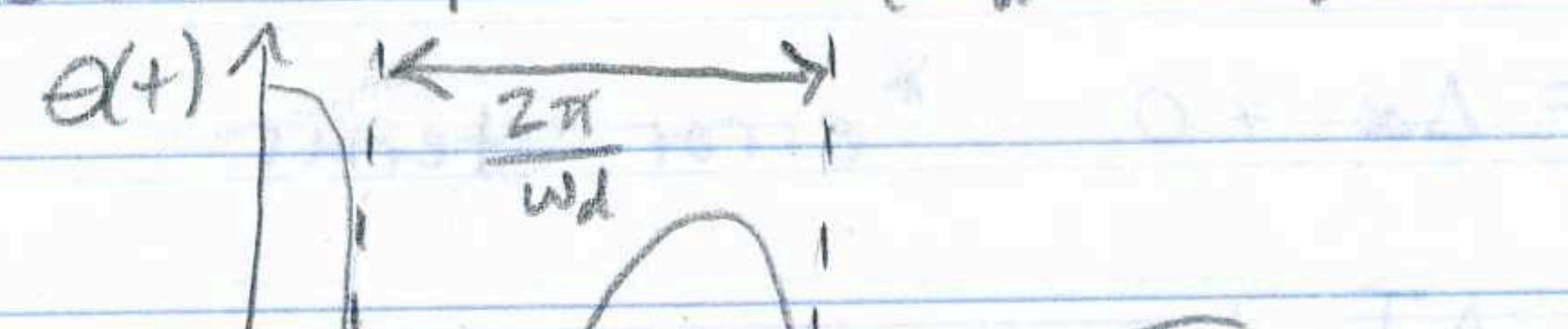


$\Theta(t) = k_1 e^{st} + k_2 t e^{st}$



underdamped: $\xi < 1$, $k_d^2 < 4Jk_p$

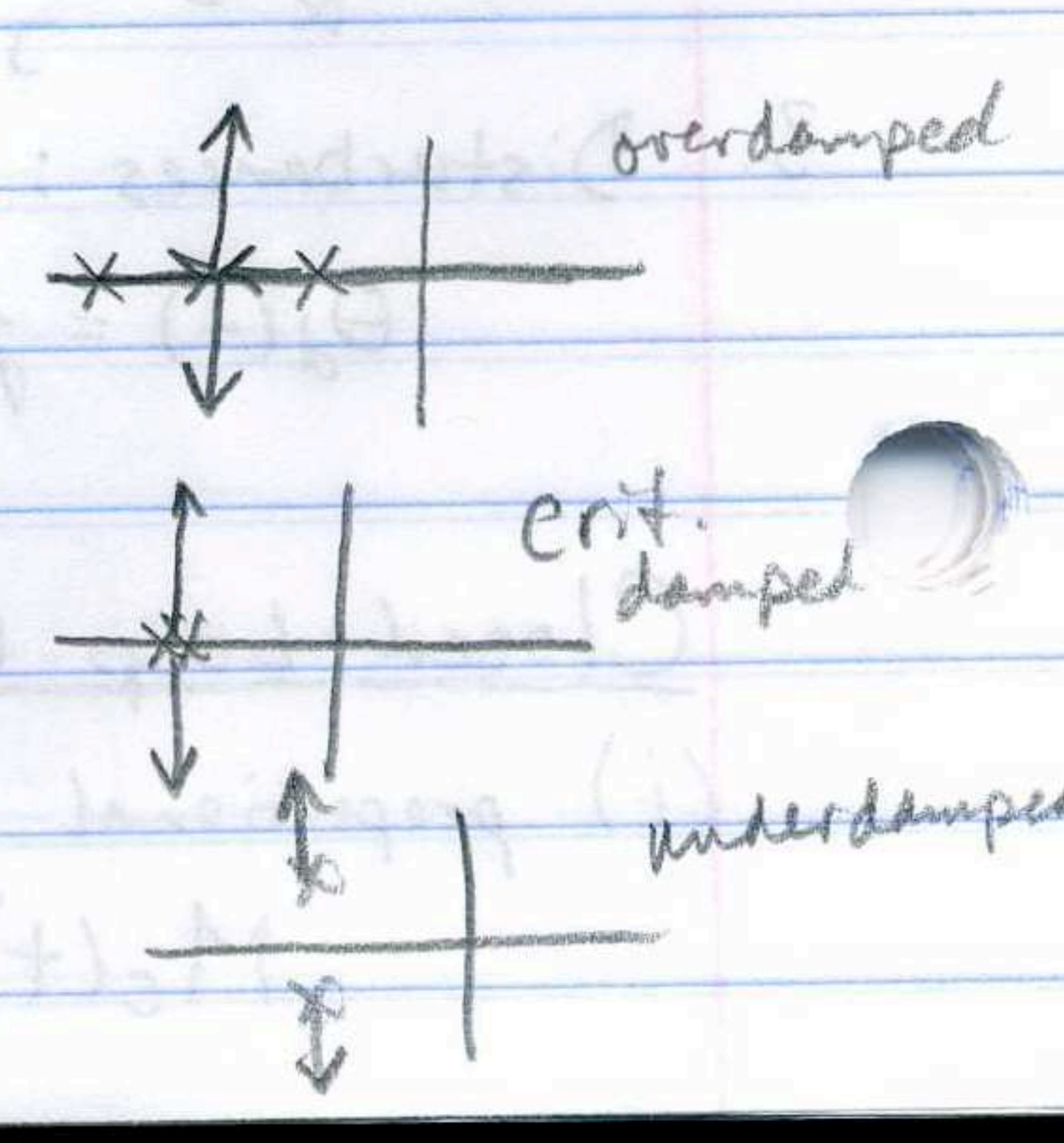
$\Theta(t) = K_1 e^{-\frac{k_d}{2}t} \sin(\omega_d t + \Theta)$

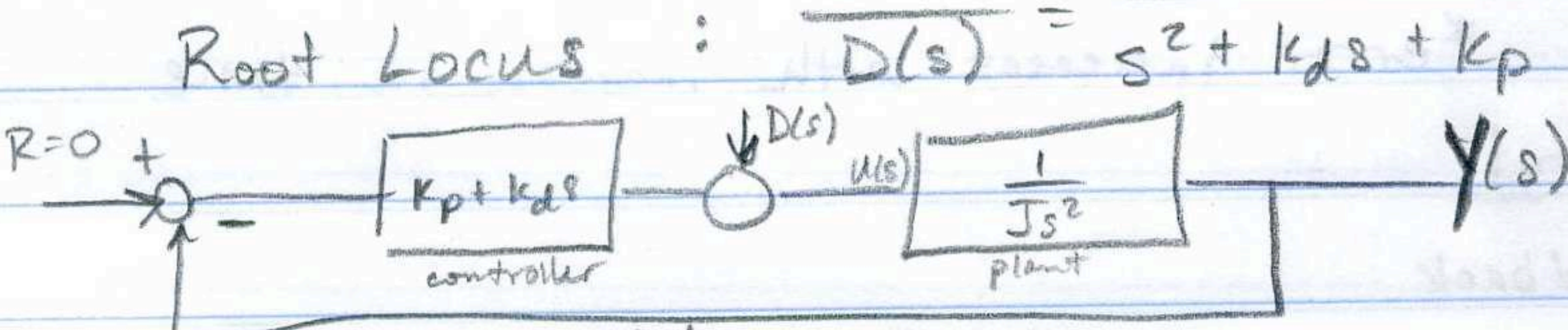


Disturbance Rejection

Assume $J = 1$, $k_d = 1$ $\quad \dfrac{\Theta(s)}{D(s)} = \dfrac{1}{s^2 + k_d s + k_p}$

Root Locus :





Assume $D = 0 \Rightarrow \dfrac{Y(s)}{R(s)} = \dfrac{k_p + k_d s}{s^2 + k_d s + k_p}$