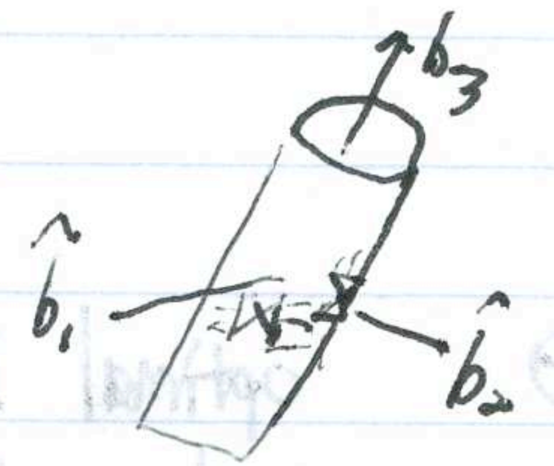
Torque-Free Axis-symmetric motion

$$0 = J_t \dot{\omega}_1 - (J_t - J_a)\,\omega_2\omega_3$$
$$0 = J_t \dot{\omega}_2 + (J_t - J_a)\,\omega_1\omega_3$$
$$0 = J_a \dot{\omega}_3$$



no longer Torque-Free:

$$M_1 = J_t\dot{\omega}_1 - (J_t - J_a)\,\omega_2\omega_3$$
$$M_2 = J_t\dot{\omega}_2 + (J_t - J_a)\,\omega_1\omega_3$$

$$\frac{M_1}{J_t} = \dot{\omega}_1 - \lambda\omega_2$$
$$\frac{M_2}{J_t} = \dot{\omega}_2 + \lambda\omega_1$$

$M_1 = 0 \quad M_2 = M\left(u(t-\tau) - u(t-(\tau+\delta))\right)$

$$M_2 = \begin{cases} 0 & t<\tau \\ M & \tau<t<\tau+\delta \\ 0 & \delta<t \end{cases}$$

$$\mathcal{L}(u(t)) = 1/s$$
$$\mathcal{L}(u(t-\tau)) = \frac{1}{s}e^{-s\tau}$$

$$\mathcal{L}(f(t)) = \int_{0^-}^{\infty} e^{-st} f(t)\,dt$$

$y = t-\tau \Rightarrow t = y+\tau \quad dy = dt \rightarrow$

$$\mathcal{L}(f(t-\tau)) = \int_{0^-}^{\infty} e^{-st} f(t-\tau)\,dt$$
$$= \int_{t=0}^{t=\infty} e^{-s(y+\tau)} f(y)\,dy$$
$$= e^{-s\tau}\underbrace{\int_{0-\tau}^{\infty} e^{-sy} f(y)\,dy}_{F(s)}$$

If both $M_1 + M_2 = 0$

$\omega_1(t) = \omega_1(0)\cos(\lambda t) + \omega_2(0)\sin(\lambda t)$

$\omega_2(t) = \omega_2(0)\cos(\lambda t) - \omega_1(0)\sin(\lambda t)$

$\dot{y} + ky = u \qquad u = \alpha_1 u_1 + \alpha_2 u_2 \rightarrow y = y_1\alpha_1 + y_2\alpha_2 + \text{IC response}$

so we don't need to deal w/ IC any more

$0 = s\omega_1 - \lambda\omega_2$

$\frac{M}{J_t}\frac{1}{s} = s\omega_2 + \lambda\omega_1 \qquad \omega_2 = \frac{1}{s}\left(-\lambda\omega_1 + \frac{M}{J_t}\frac{1}{s}\right)$

$$0 = s\omega_1 - \frac{\lambda}{s}\left(-\lambda\omega_1 + \frac{M}{J_t}\frac{1}{s}\right) = s^2\omega_1 + \lambda^2\omega_1 - \frac{\lambda M}{J_t}\frac{1}{s}$$

$$\omega_1 = \frac{\lambda M}{J_t}\frac{1}{s(s^2+\lambda^2)} = \frac{\lambda M}{J_t}\left(\frac{a}{s} + \frac{sb}{s^2+\lambda^2}\right) \rightarrow \frac{as^2 + a\lambda^2 + bs^2}{s(s^2+\lambda^2)} \qquad a = -b \quad a = 1/\lambda^2$$

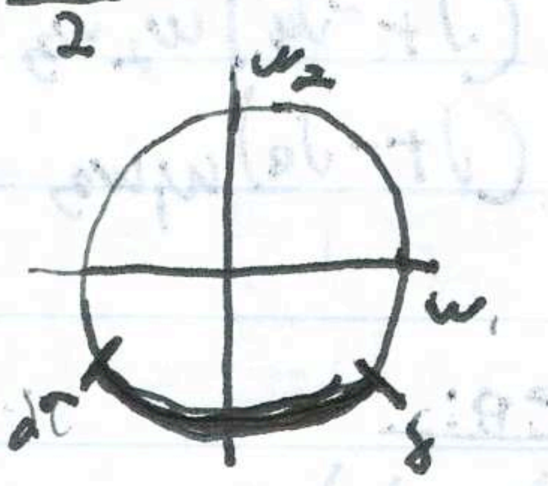
$$\omega_1 = \frac{M}{\lambda J_t}\left(\frac{1}{s} - \frac{s}{s^2+\lambda^2}\right) \quad \text{"}\omega_1(t) = \frac{M}{\lambda J_t}\left(u(t) - \cos(\lambda t)\right)\text{"}$$

$$\boxed{\omega_1(t) = \omega_1(0)\cos(\lambda t) + \omega_2(0)\sin(\lambda t) + \frac{M}{\lambda J_t}\left(u(t-\tau) - \cos(\lambda(t-\tau))\right) - \frac{M}{\lambda J_t}\left(u(t-(\tau+\delta)) - \cos(\lambda(t-(\tau+\delta)))\right)}$$

Unit steps cancel $\omega_2(0) = 0 \rightarrow$

$$\boxed{\begin{aligned} \omega_1(t) &= \omega_1(0)\cos\lambda t + \frac{M}{\lambda J_t}\left(-\cos(\lambda(t-\tau)) + \cos(\lambda(t-(\tau+\delta)))\right) \\ \omega_2(t) &= -\omega_1(0)\sin\lambda t + \frac{M}{\lambda J_t}\left(\sin(\lambda(t-\tau)) - \sin(\lambda(t-(\tau+\delta)))\right) \end{aligned}}$$

$\omega_t^2 = \omega_1^2 + \omega_2^2$ $\qquad \frac{d(\omega_t^2)}{d\tau} = 0 \to \lambda\tau = \frac{\pi}{2} - \frac{\lambda\delta}{2}$



centered around $-\omega_2$

$\tan\theta - \frac{J_t \omega_t \Delta}{J_a \lambda} \approx \Delta\theta$

optimal $\lambda\delta = \pi$
pulse width