AE403: Alternative Ways of Transforming a Rotation Matrix into a Quaternion

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In class we saw a way to construct the quaternion

$$Q = iq_1 + jq_2 + kq_3 + q_4$$

that corresponds to a given rotation matrix

$$R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}.$$

The method we discussed worked only when $q_4 \neq 0$. There are three other ways to construct Q that work when $q_4 = 0$. These methods are as follows:

1. Compute q_1 first. By convention, we choose the sign of q_1 that makes $q_4 \geq 0$.

$$\begin{aligned} q_1 &= \pm\frac{1}{2}\sqrt{1 + R_{11} - R_{22} - R_{33}} \\ q_2 &= \frac{1}{4q_1}(R_{12} + R_{21}) \\ q_3 &= \frac{1}{4q_1}(R_{13} + R_{31}) \\ q_4 &= \frac{1}{4q_1}(R_{23} - R_{32}) \end{aligned}$$

2. Compute q_2 first. By convention, we choose the sign of q_2 that makes $q_4 \geq 0$.

$$\begin{aligned} q_2 &= \pm\frac{1}{2}\sqrt{1 - R_{11} + R_{22} - R_{33}} \\ q_1 &= \frac{1}{4q_2}(R_{12} + R_{21}) \\ q_3 &= \frac{1}{4q_2}(R_{23} + R_{32}) \\ q_4 &= \frac{1}{4q_2}(R_{31} - R_{13}) \end{aligned}$$

3. Compute q_3 first:

$$q_3 = \pm\frac{1}{2}\sqrt{1 - R_{11} - R_{22} + R_{33}}$$
$$q_1 = \frac{1}{4q_3}(R_{31} + R_{13})$$
$$q_2 = \frac{1}{4q_3}(R_{23} + R_{32})$$
$$q_4 = \frac{1}{4q_3}(R_{12} - R_{21})$$

By convention, we choose the sign of q_1 that makes $q_4 \geq 0$.

There are some things to understand about these alternative methods. First, all of them will give you the same answer. Second, at least one of them will always work—this is because $q_1^2 + q_2^2 + q_3^2 + q_4^2$, so some q_i must satisfy

$$|q_i| \geq \frac{1}{2}.$$

Third, it is not just when $q_4 = 0$ that we should choose an alternative method. When q_4 gets very small, we may begin to have numerical problems. So in general, we should choose the method that corresponds to the largest $|q_i|$ in order to get good numerical accuracy.