2/7/07

Quaternions: $Q = iq_1 + jq_2 + kq_3 + q_4$

$$i^2 = j^2 = k^2 = -1 \; ; \; ij = k, \; jk = i, \; ki = j$$

$$ji = -k, \; kj = -i, \; ik = -j$$

Equivalent axis: $\hat{e} = e_1 \hat{a}_1 + e_2 \hat{a}_2 + e_3 \hat{a}_3$

$$= e_1 \hat{b}_1 + e_2 \hat{b}_2 + e_3 \hat{b}_3$$

Equivalent angle: $\mu$

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$q_1 = e_1 \sin(\mu/2), \quad q_2 = e_2 \sin(\mu/2) \quad \leftarrow q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$

$q_3 = e_3 \sin(\mu/2), \quad q_4 = \cos(\mu/2)$

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$$|Q|^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2$$

$$= \sin^2(\mu/2)\cancelto{1}{(e_1^2 + e_2^2 + e_3^2)} + \cos^2(\mu/2) = 1$$

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$$q_1 = -e_1 \sin(-\mu/2) = e_1 \sin(\mu/2)$$

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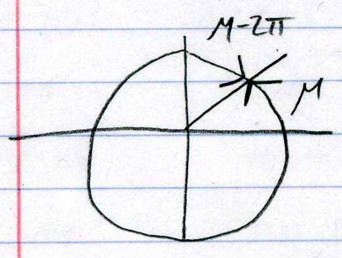


$$q_1 = e_1 \sin\left(\frac{\mu - 2\pi}{2}\right) = -e_1 \sin(\mu/2)$$

$$q_2 = -e_2 \sin(\mu/2)$$

$$q_3 = -e_3 \sin(\mu/2)$$

$$q_4 = -\cos(\mu/2)$$

$Q$ and $-Q$ are the same rotation

Find $\hat{e}$ and $\mu$ given $Q$

$$\boxed{\mu = 2\cos^{-1}(q_4)} \quad (0 \leq \mu \leq \pi)$$

$$q_1^2 + q_2^2 + q_3^2 = \sin^2\left(\frac{\mu}{2}\right)$$

$$\sin(\mu/2) = \sqrt{q_1^2 + q_2^2 + q_3^2} \qquad \text{(positive by convention)}$$

$$e_1 = q_1 / \sqrt{q_1^2 + q_2^2 + q_3^2}$$

$$\boxed{\hat{e} = \frac{q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}}$$

## Q → Rotation Matrix

$$R(q, q_4) = (q_4^2 - q^T q) I + 2qq^T - 2q_4 S(q)$$

$$S(q) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ q_2 & q_1 & 0 \end{bmatrix}$$

$$R(q, q_4) = \begin{bmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2+q_3q_4) & 2(q_1q_3-q_2q_4) \\ 2(q_1q_2-q_3q_4) & 1-2(q_1^2+q_3^2) & 2(q_2q_3+q_1q_4) \\ 2(q_1q_3+q_2q_4) & 2(q_2q_3-q_1q_4) & 1-2(q_1^2+q_2^2) \end{bmatrix}$$

## R → Q

$$R_{11} + R_{22} + R_{33} = 3 - 4(q_1^2 + q_2^2 + q_3^2) = 3 - 4(1 - q_4^2)$$

$$= 4q_4^2 - 1$$

$$q_4^2 = \pm\frac{1}{2}\sqrt{1 + R_{11} + R_{22} + R_{33}}$$

If $q_4 \neq 0$:

$R_{12} - R_{21} = 4 q_3 q_4$ $\qquad q_3 = \frac{1}{4q_4}(R_{12} - R_{21})$

$$\boxed{q = \frac{1}{4q_4}\begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix}}$$