

2/7/07

Quaternions: $Q = iq_1 + jq_2 + kq_3 + q_4$

$$i^2 = j^2 = k^2 = -1; ij = k, jk = i, ki = j$$

$$ji = -k, kj = -i, ik = -j$$

Equivalent axis: $\hat{e} = e_1 \hat{a}_1 + e_2 \hat{a}_2 + e_3 \hat{a}_3$
 $= e_1 \hat{b}_1 + e_2 \hat{b}_2 + e_3 \hat{b}_3$

Equivalent angle: M

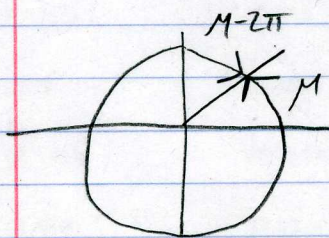
$$q_1 = e_1 \sin(M/2), q_2 = e_2 \sin(M/2) \leftarrow q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$q_3 = e_3 \sin(M/2), q_4 = \cos(M/2)$$

$$|Q|^2 = q_1^2 + q_2^2 + q_3^2 + q_4^2$$

$$= \sin^2(M/2) (e_1^2 + e_2^2 + e_3^2) + \cos^2(M/2) = 1$$

$$q_1 = -e_1 \sin(-M/2) = e_1 \sin(M/2)$$



$$q_1 = e_1 \sin\left(\frac{M-2\pi}{2}\right) = -e_1 \sin(M/2)$$
$$q_2 = -e_2 \sin(M/2)$$
$$q_3 = -e_3 \sin(M/2)$$
$$q_4 = -\cos(M/2)$$

Q and $-Q$ are the same rotation

Find \hat{e} and M given Q

$$M = 2 \cos^{-1}(q_4) \quad (0 \leq M \leq \pi)$$

$$q_1^2 + q_2^2 + q_3^2 = \sin^2(M/2)$$

$$\sin(M/2) = \sqrt{q_1^2 + q_2^2 + q_3^2} \quad (\text{positive by convention})$$

$$e_1 = q_1 / \sqrt{q_1^2 + q_2^2 + q_3^2}$$

$$\hat{e} = \frac{q_1 \hat{a}_1 + q_2 \hat{a}_2 + q_3 \hat{a}_3}{\sqrt{q_1^2 + q_2^2 + q_3^2}}$$

$Q \rightarrow$ Rotation Matrix

$$R(q, q_4) = (q_4^2 - q^T q) I + 2 q q^T - 2 q_4 S(q)$$

$$S(q) = \begin{bmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ q_2 & q_1 & 0 \end{bmatrix}$$

$$R(q, q_4) = \begin{bmatrix} 1 - 2(q_2^2 + q_3^2) & 2(q_1 q_2 + q_3 q_4) & 2(q_1 q_3 - q_2 q_4) \\ 2(q_1 q_2 - q_3 q_4) & 1 - 2(q_1^2 + q_3^2) & 2(q_2 q_3 + q_1 q_4) \\ 2(q_1 q_3 + q_2 q_4) & 2(q_2 q_3 - q_1 q_4) & 1 - 2(q_1^2 + q_2^2) \end{bmatrix}$$

$R \rightarrow Q$

$$\begin{aligned} R_{11} + R_{22} + R_{33} &= 3 - 4(q_1^2 + q_2^2 + q_3^2) = 3 - 4(1 - q_4^2) \\ &= 4q_4^2 - 1 \end{aligned}$$

$$q_4^2 = \pm \frac{1}{2} \sqrt{1 + R_{11} + R_{22} + R_{33}}$$

If $q_4 \neq 0$:

$$R_{12} - R_{21} = 4q_3q_4 \quad q_3 = \frac{1}{4q_4}(R_{12} - R_{21})$$

$$q = \frac{1}{4q_4} \begin{bmatrix} R_{23} - R_{32} \\ R_{31} - R_{13} \\ R_{12} - R_{21} \end{bmatrix}$$

$(\frac{1}{\sqrt{2}})^T \hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2)$
 $(\frac{1}{\sqrt{2}})^T \hat{a} = \frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2) = (\frac{1}{\sqrt{2}})^T \hat{a}$
 $\frac{1}{\sqrt{2}}(\hat{a}_1 + \hat{a}_2) = \hat{a}$
 $\hat{a}_1 + \hat{a}_2 = \sqrt{2}\hat{a}$

$\hat{a} \rightarrow$ Rotation Matrix

$$R(\hat{a}, \hat{a}) = (\hat{a}_1 - \hat{a}_2) \hat{a}$$

$$R(\hat{a}) = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & 0 \\ 0 & 0 & \hat{a}_1 \\ 0 & \hat{a}_1 & \hat{a}_2 \end{bmatrix}$$

$$R(\hat{a}, \hat{a}) = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & 0 \\ 0 & 0 & \hat{a}_1 \\ 0 & \hat{a}_1 & \hat{a}_2 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & 0 \\ 0 & 0 & \hat{a}_1 \\ 0 & \hat{a}_1 & \hat{a}_2 \end{bmatrix}$$

$\hat{a} \rightarrow \hat{a}$

$$R_{11} + R_{22} + R_{33} = 3 - N(\hat{a}_1^2 + \hat{a}_2^2) = 3 - N(1 - \hat{a}_4^2)$$

$$= N\hat{a}_4^2 - 1$$

$$\hat{a}_4^2 = \frac{1}{2} \sqrt{1 + R_{11} + R_{22} + R_{33}}$$