AE403: Direction Cosine Matrices

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Euler angles are one way to construct a rotation matrix. *Direction cosines* are another way. Let R be the rotation matrix taking frame A to frame B, so

$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = R \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}.$$

Write R in terms of its components R_{ij}, so we have

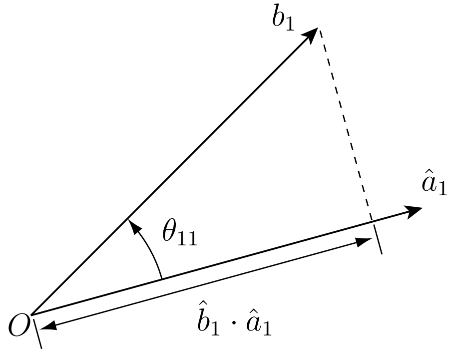
$$\begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}.$$

Using the first row of R, we can express the unit vector $\hat{b}_1$ in the coordinates of frame A as

$$\hat{b}_1 = R_{11}\hat{a}_1 + R_{12}\hat{a}_2 + R_{13}\hat{a}_3.$$

What this says is that R_{11} is "the amount that $\hat{b}_1$ extends along the $\hat{a}_1$ direction." In other words, $R_{11} = \hat{b}_1 \cdot \hat{a}_1$. And by the definition of the dot product, we have

$$\begin{aligned} R_{11} &= |\hat{b}_1||\hat{a}_1| \cos\theta_{11} \\ &= (1)(1)\cos\theta_{11} \\ &= \cos\theta_{11} \end{aligned}$$



where θ_{11} is the angle between $\hat{b}_1$ and $\hat{a}_1$. Doing the same for every entry R_{ij}, we write

$$\begin{aligned}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix} &= \begin{bmatrix}\hat{b}_1\cdot\hat{a}_1 & \hat{b}_1\cdot\hat{a}_2 & \hat{b}_1\cdot\hat{a}_3\\ \hat{b}_2\cdot\hat{a}_1 & \hat{b}_2\cdot\hat{a}_2 & \hat{b}_2\cdot\hat{a}_3\\ \hat{b}_3\cdot\hat{a}_1 & \hat{b}_3\cdot\hat{a}_2 & \hat{b}_3\cdot\hat{a}_3\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta_{11} & \cos\theta_{12} & \cos\theta_{13}\\ \cos\theta_{21} & \cos\theta_{22} & \cos\theta_{23}\\ \cos\theta_{31} & \cos\theta_{32} & \cos\theta_{33}\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{bmatrix}\end{aligned}$$

This is a *direction cosine matrix*. Each entry is the cosine of the angle between some unit vector of B and some unit vector of A. Notice that every direction cosine matrix is a rotation matrix, and every rotation matrix can be written in terms of direction cosines—it is not even fair to distinguish these two matrices, they are really the same thing.

The important thing to realize is that these nine numbers—whether you think of them as direction cosines or just as elements of a rotation matrix—are another way to parameterize orientation. This representation has no *singularities* as did the Euler angle representation. We will see next week that the relationship between each rate of change dR_{ij}/dt and the angular velocity components $\omega_1, \omega_2, \omega_3$ is always well-defined. However, this representation is subject to *constraints* because, as we've discussed before, not every matrix is a rotation matrix.

For your homework, it is useful to know one more thing. Notice that row i of the direction cosine matrix gives the coordinates of $\hat{b}_i$ in frame A. For example, the coordinates of $\hat{b}_1$ are

$$\begin{bmatrix}R_{11} & R_{12} & R_{13}\end{bmatrix}^T$$

in frame A, as we said before. This means that direction cosine matrices (equivalently, rotation matrices) are often easy to write by inspection. For example, let's say frame B was oriented such that axis $\hat{b}_1$ is aligned with $-\hat{a}_2$, axis $\hat{b}_2$ is aligned with $\hat{a}_1$, and $\hat{b}_3$ is aligned with $\hat{a}_3$. Then the coordinates of $\hat{b}_1$ are

$$\begin{bmatrix}0 & -1 & 0\end{bmatrix}^T$$

and so on, so we can immediately write

$$R = \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}.$$