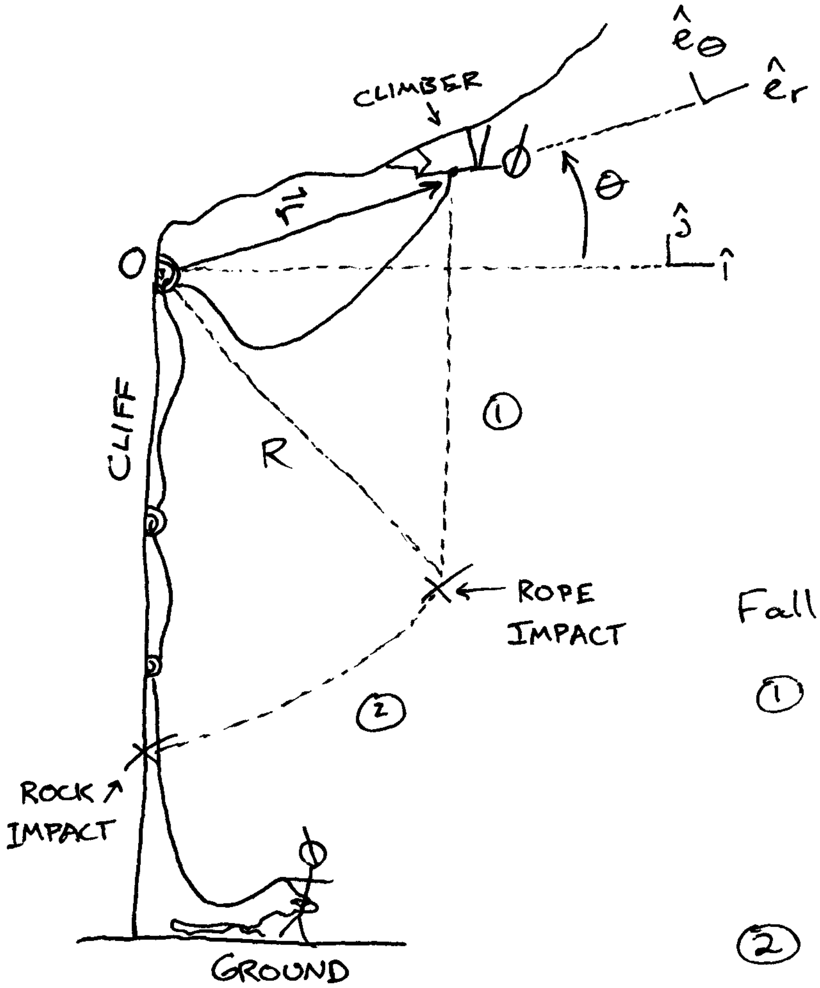
AE 252 (Spring 2007) 1/23/07

Bretl

Example: polar coordinates



A rope leads through anchors to the climber, to protect against falls. Location of climber is $\vec{r} = r\hat{e}_r$. (Initially, $r = r_0$ and $\theta = \theta_0$.)

Fall occurs in two stages:

① Total acceleration $\vec{a} = -g\hat{\jmath}$ until $r = R \Rightarrow$ tight rope ("rope impact")

② Transverse acceleration $a_\theta = -g\cos\theta$ until $\theta = -\frac{\pi}{2}$ ("rock impact")

In practice, $R - r_0 > 0$ (in other words, there is extra slack in the system). In fact, this extra slack is critical for safety. Why???

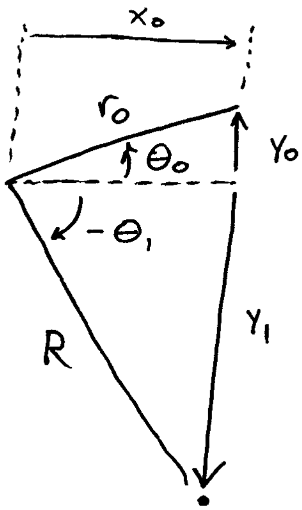
FIND...

related to energy at impact... the lower the better!

$$\begin{cases} V_r^2 & \text{at rope impact} \\ V_\theta^2 & \text{at rock impact} \end{cases}$$

as a function of R (given r_0, θ_0, g).

(1)



$$x_o = r_o \cos\theta_o$$
$$y_o = r_o \sin\theta_o$$

$$y_1 = \sqrt{R^2 - r_o^2\cos^2\theta_o} = \sqrt{R^2 - x_o^2}$$

Want to find $\vec{v}$ at rope impact.

STRAIGHT-LINE MOTION, CONSTANT ACCELERATION!

$$\left.\begin{aligned}\vec{v} &= v\hat{j}\\ \vec{a} &= -g\hat{j}\end{aligned}\right\} \quad \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy} = -g$$

Integrate: $\displaystyle\int_{v_o}^{v} v\,dv = \int_{y_o}^{y} -g\,dy$

$$\frac{v^2}{2} - \cancelto{0}{\frac{v_o^2}{2}} = -g(y - y_o)$$

$$\Rightarrow \quad v^2 = -2g(y - y_o)$$

At $y = -y_1$, $\quad \vec{v} = -\sqrt{2g(y_o + y_1)}\ \hat{j}$.

Put this in polar coordinates:

$$\vec{v} = -\sin\theta_1\sqrt{2g(y_o+y_1)}\ \hat{e}_r - \cos\theta_1\sqrt{2g(y_o+y_1)}$$

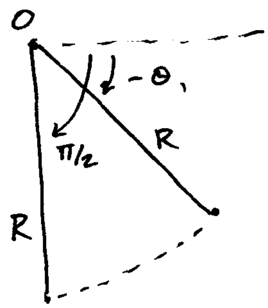
$$\sin\theta_1 = -\frac{y_1}{R}, \quad \cos\theta_1 = \frac{x_o}{R}$$

$$\Rightarrow \quad \vec{v} = \frac{y_1}{R}\sqrt{2g(y_o+y_1)}\ \hat{e}_r - \frac{x_o}{R}\sqrt{2g(y_o+y_1)}\ \hat{e}_\theta$$

So at rope impact,

$$\boxed{v_r^2 = \frac{y_1^2}{R^2}\left(2g(y_o + y_1)\right)}$$

②



Want to find $\vec{v}$ at rock impact.

$$\vec{v} = \cancel{\dot{r}}^{\,0}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$$= R\dot{\theta}\,\hat{e}_\theta$$

So we need to find $\dot{\theta}$ at impact.

We are given $a_\theta = -g\cos\theta$ $\quad (\vec{a} = a_r\hat{e}_r + a_\theta\hat{e}_\theta)$.

In polar coordinates, $a_\theta = r\ddot{\theta} + 2\cancel{\dot{r}}^{\,0}\dot{\theta} = R\ddot{\theta}$.

Let $\omega = \dot{\theta}$. Then $R\ddot{\theta} = R\frac{d\omega}{dt} = R\omega\frac{d\omega}{d\theta} = -g\cos\theta$.

Integrate: $\displaystyle\int_{\omega_1}^{\omega} \omega\, d\omega = -\frac{g}{R}\int_{\theta_1}^{\theta} \cos\theta\, d\theta$

$$\frac{\omega^2 - \omega_1^2}{2} = -\frac{g}{R}(\sin\theta - \sin\theta_1)$$

$$\Rightarrow \quad \dot{\theta}^2 = \dot{\theta}_1^2 - \frac{2g}{R}(\sin\theta - \sin\theta_1).$$

Already found $\sin\theta_1 = -y_1/R$. Know $\sin(-\frac{\pi}{2}) = -1$.

Still need $\dot{\theta}_1$ — we found

$$V_\theta = -\frac{x_0}{R}\sqrt{2g(y_0+y_1)} \quad \text{at} \quad \theta = \theta_1$$

$$= R\dot{\theta}_1 \quad \Rightarrow \quad \dot{\theta}_1 = -\frac{x_0}{R^2}\sqrt{2g(y_0+y_1)}$$

So at rock impact,

$$\boxed{V_\theta^2 = R^2\dot{\theta}^2 = \frac{x_0^2}{R^2}\left(2g(y_0+y_1)\right) + 2gR\left(1 - \frac{y_1}{R}\right)}$$

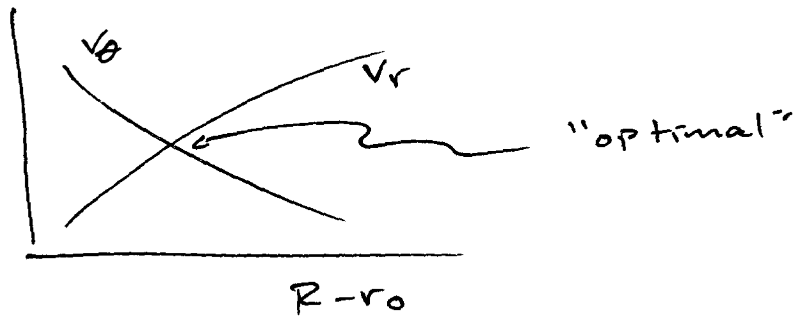
So what?

- For given r_0, θ_0, g

  plot v_r^2 and v_θ^2 as functions

  of $R - r_0$ (of extra slack)

- Note $v_r \uparrow$ and $v_\theta \downarrow$ as $(R-r_0) \uparrow$



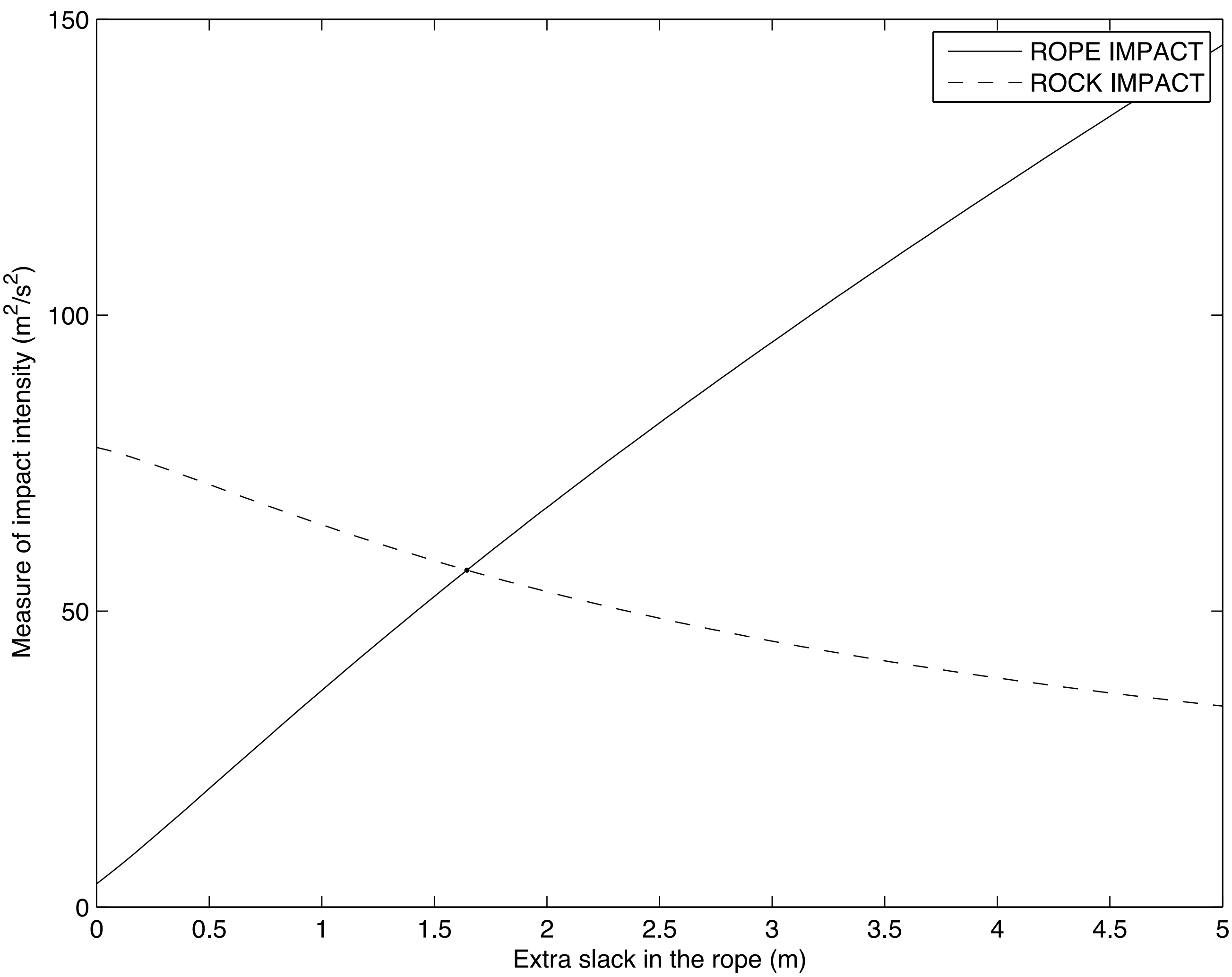
(not intuitive to non-climbers)

- Optimal is <u>not</u> <u>zero</u>

- Optimal <u>increases</u> as you climb farther

  (not intuitive to climbers)

Conclusion ... this is a dangerous sport!


ROPE IMPACT
ROCK IMPACT
Measure of impact intensity (m^2/s^2)
Extra slack in the rope (m)
0
50
100
150
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5