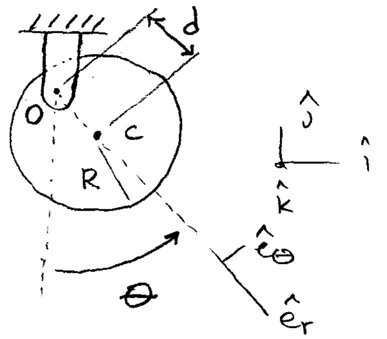
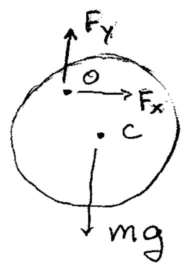
Moments about points other than C

We have been using $\sum \vec{M}_C = \frac{{}^{\mathcal{F}}d\vec{H}_C}{dt}$. For planar motion, this expression is very simple:

$$\sum \vec{M}_C = I_C \alpha_B \hat{k} .$$

Sometimes it is even easier to take moments about other points. For example:





Here, if we take moments about C, we will also have to consider translational dynamics to get rid of F_x and F_y:

$$\sum \vec{M}_C = d(-\sin\theta\, \hat{i} + \cos\theta\, \hat{j}) \times (F_x \hat{i} + F_y \hat{j})$$

$$= -d(F_x \cos\theta + F_y \sin\theta)\, \hat{k}$$

$${}^{\mathcal{F}}\vec{\alpha}^B = \ddot{\theta}\, \hat{k} , \quad I_C = mR^2/2$$

$$\Rightarrow \boxed{-d(F_x \cos\theta + F_y \sin\theta) = \frac{mR^2}{2}\ddot{\theta}}$$

unknown!

Since F_x, F_y are unknown, we also need

$$\Sigma \vec{F} = F_x \hat{\imath} + (F_y - mg)\hat{\jmath}$$

$$\vec{a}_c = -d\dot{\theta}^2 \hat{e}_r + d\ddot{\theta}\hat{e}_\theta$$

$\hat{e}_r = -\cos\theta\hat{\jmath} + \sin\theta\hat{\imath}$

$\hat{e}_\theta = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$

$$= (-d\dot{\theta}^2\sin\theta + d\ddot{\theta}\cos\theta)\hat{\imath} + (d\dot{\theta}^2\cos\theta + d\ddot{\theta}\sin\theta)\hat{\jmath}$$

$$\Rightarrow \boxed{\begin{aligned} F_x &= (-d\dot{\theta}^2\sin\theta + d\ddot{\theta}\cos\theta)m \\ F_y - mg &= (d\dot{\theta}^2\cos\theta + d\ddot{\theta}\sin\theta)m \end{aligned}}$$

Say we had instead taken moments about O:

$$\Sigma \vec{M}_O = d(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \times (-mg\hat{\jmath})$$

$$= -mgd\sin\theta\,\hat{k}$$

What is the rate of change of angular momentum?

Recall $\quad \vec{H}_O = \vec{H}_c + \vec{r}_c \times m\,{}^{F}\vec{v}_c$

$$\Rightarrow \frac{{}^{F}d\vec{H}_O}{dt} = \frac{{}^{F}d\vec{H}_c}{dt} + \underbrace{\cancel{\vec{v}_c \times m\,{}^{F}\vec{v}_c}}_{0} + \vec{r}_c \times m\,{}^{F}\vec{a}_c$$

Since <u>both</u> O and C are fixed in B,

$${}^{F}\vec{a}_c = \underbrace{\cancel{{}^{F}\vec{a}_O}}_{0} + {}^{F}\vec{\alpha}^{B} \times \vec{r}_c + {}^{F}\vec{\omega}^{B} \times ({}^{F}\vec{\omega}^{B} \times \vec{r}_c)$$

$$= \ddot{\theta}\hat{k} \times d\hat{e}_r + \dot{\theta}\hat{k} \times (\dot{\theta}\hat{k} \times d\hat{e}_r)$$

$$= d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_r$$

$$\Rightarrow \vec{r}_c \times m\,{}^{F}\vec{a}_c = d\hat{e}_r \times m(d\ddot{\theta}\hat{e}_\theta - d\dot{\theta}^2\hat{e}_r)$$

$$= (md^2)\ddot{\theta}\hat{k}$$

So in fact, we find

$$\frac{{}^{\mathcal{F}}d\vec{H}_O}{dt} = \frac{{}^{\mathcal{F}}d\vec{H}_C}{dt} + (md^2)\ddot{\theta}\,\hat{k}$$

$$= \underbrace{\left(I_C + md^2\right)}\,\ddot{\theta}\,\hat{k}$$

We call this I_O, the mass moment of inertia of a body about any other point O that is fixed in the body:

$$\boxed{I_O = I_C + md^2}$$

"parallel axis theorem"

To finish our example, we find

$$-mgd\sin\theta\,\hat{k} = (I_C + md^2)\,\ddot{\theta}\,\hat{k}$$

$$\Rightarrow \boxed{-mgd\sin\theta = \left(\frac{mR^2}{2} + md^2\right)\ddot{\theta}}$$

This is a complete description of the disk's motion, found without having to consider F_x and F_y.

Two things were very important in this derivation. First, O has to be fixed in an inertial reference frame. Otherwise, we could not assume

$$\sum \vec{M}_O = {}^{F}\frac{d\vec{H}_O}{dt}.$$

(Instead, we would have extra terms.)
Second, O also has to be fixed in the rigid body B. Otherwise, we could not assume

$${}^{F}\frac{d\vec{H}_O}{dt} = I_O \alpha_B \hat{k}.$$

(Again, we would have extra/different terms.)

Text problem 18.67 is a good example!