LAST TIME

Continued to use the expression

$$\boxed{{}^{\mathcal{F}}\frac{d\vec{u}}{dt} = {}^{B}\frac{d\vec{u}}{dt} + {}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{r}_{B/A}}$$

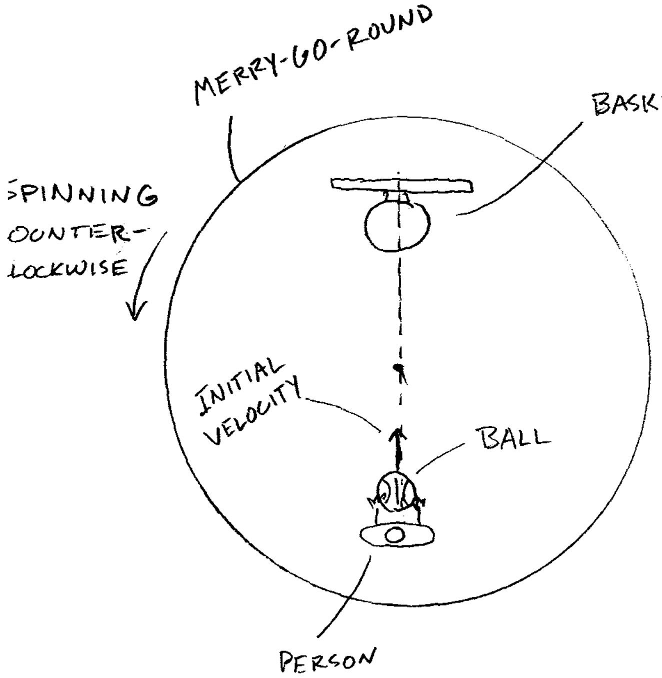
to find the time rate of change of <u>any</u> vector $\vec{u}$ as <u>observed</u> from frame $\mathcal{F}$, when $\vec{u}$ itself is <u>described</u> in the coordinates of frame B.

In particular, we used this expression to relate the velocities and accelerations of <u>any</u> two points P and Q, when the <u>relative</u> <u>position</u> $\vec{r}_{Q/P}$ between them is <u>described</u> in the coordinates of some moving reference frame B, which we think of as <u>attached</u> to a rigid body.

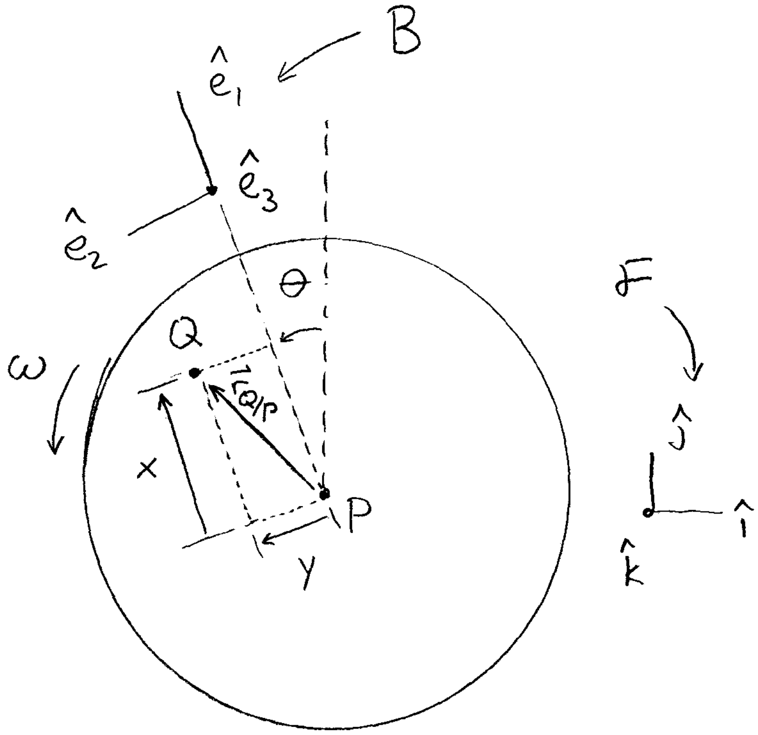
$$
\boxed{
\begin{aligned}
{}^{\mathcal{F}}\vec{V}_Q &= {}^{\mathcal{F}}\vec{V}_P + \boxed{{}^{B}\vec{V}_{Q/P}} + {}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{r}_{Q/P} \\
{}^{\mathcal{F}}\vec{a}_Q &= {}^{\mathcal{F}}\vec{a}_P + \boxed{{}^{B}\vec{a}_{Q/P}} + {}^{\mathcal{F}}\vec{\alpha}^{B} \times \vec{r}_{Q/P} + \boxed{2\,{}^{\mathcal{F}}\vec{\omega}^{B} \times {}^{B}\vec{V}_{Q/P}} + {}^{\mathcal{F}}\vec{\omega}^{B} \times \left({}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{r}_{Q/P}\right)
\end{aligned}
}
$$

These terms are "new" — when points P and Q are not <u>fixed</u> in a rigid body, then Q <u>moves</u> relative to P in the body, so ${}^{B}\vec{V}_{Q/P}$ and ${}^{B}\vec{a}_{Q/P}$ are <u>not</u> <u>zero</u>.

As an example, we started talking about playing basketball on a merry-go-round.



Intuitive picture



model

In our model...

P = center of merry-go-round

Q = basketball

$\vec{r}_{Q/P}$ = position of ball relative to center

$$\vec{r}_{Q/P} = x\hat{e}_1 + y\hat{e}_2$$

$$\left.\begin{aligned} \vec{r}_{Q/P}(t=0) &= -D\hat{e}_1 \\ {}^{B}\vec{v}_{Q/P}(t=0) &= V_0\hat{e}_1 \end{aligned}\right\}$$ initial position + velocity of ball

$\vec{r}_{Q/P}(\text{basket}) = D\hat{e}_1$ — position of basket

Find velocity and acceleration of ball.

Method 1

$$^{\mathcal{F}}\vec{v}_Q = {}^{\mathcal{F}}\vec{v}_P + {}^{B}\vec{v}_{Q/P} + {}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{r}_{Q/P}$$

$$= 0 + (\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2) + \omega\hat{k} \times (x\hat{e}_1 + y\hat{e}_2)$$

$$\Rightarrow \boxed{^{\mathcal{F}}\vec{v}_Q = (\dot{x} - \omega y)\hat{e}_1 + (\dot{y} + \omega x)\hat{e}_2}$$

$$^{\mathcal{F}}\vec{a}_Q = {}^{\mathcal{F}}\vec{a}_P + {}^{B}\vec{a}_{Q/P} + {}^{\mathcal{F}}\vec{\alpha}^{B} \times \vec{r}_{Q/P} + 2\,{}^{\mathcal{F}}\vec{\omega}^{B} \times {}^{B}\vec{v}_{Q/P} + {}^{\mathcal{F}}\vec{\omega}^{B} \times ({}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{r}_{Q/P})$$

$$^{\mathcal{F}}\vec{a}_P = 0$$

$$^{B}\vec{a}_{Q/P} = \ddot{x}\hat{e}_1 + \ddot{y}\hat{e}_2$$

$$^{\mathcal{F}}\vec{\alpha}^{B} \times \vec{r}_{Q/P} = 0 \times \vec{r}_{Q/P} = 0$$

$$2\,{}^{\mathcal{F}}\vec{\omega}^{B} \times {}^{B}\vec{v}_{Q/P} = 2\omega\hat{k} \times (\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2) = -2\omega\dot{y}\hat{e}_1 + 2\omega\dot{x}\hat{e}_2$$

$$^{\mathcal{F}}\vec{\omega}^{B} \times ({}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{r}_{Q/P}) = \omega\hat{k} \times (\omega\hat{k} \times (x\hat{e}_1 + y\hat{e}_2))$$

$$= \omega\hat{k} \times (\omega x\hat{e}_2 - \omega y\hat{e}_1)$$

$$= -\omega^2 x\,\hat{e}_1 - \omega^2 y\,\hat{e}_2$$

$$\Rightarrow \boxed{^{\mathcal{F}}\vec{a}_Q = (\ddot{x} - 2\omega\dot{y} - \omega^2 x)\hat{e}_1 + (\ddot{y} + 2\omega\dot{x} - \omega^2 y)\hat{e}_2}$$

We can use ${}^{\mathcal{F}}\vec{a}_Q$ to find the equations of motion for the ball (Newton's Laws):

$$\sum \vec{F} = m_Q\, {}^{\mathcal{F}}\vec{a}_Q$$

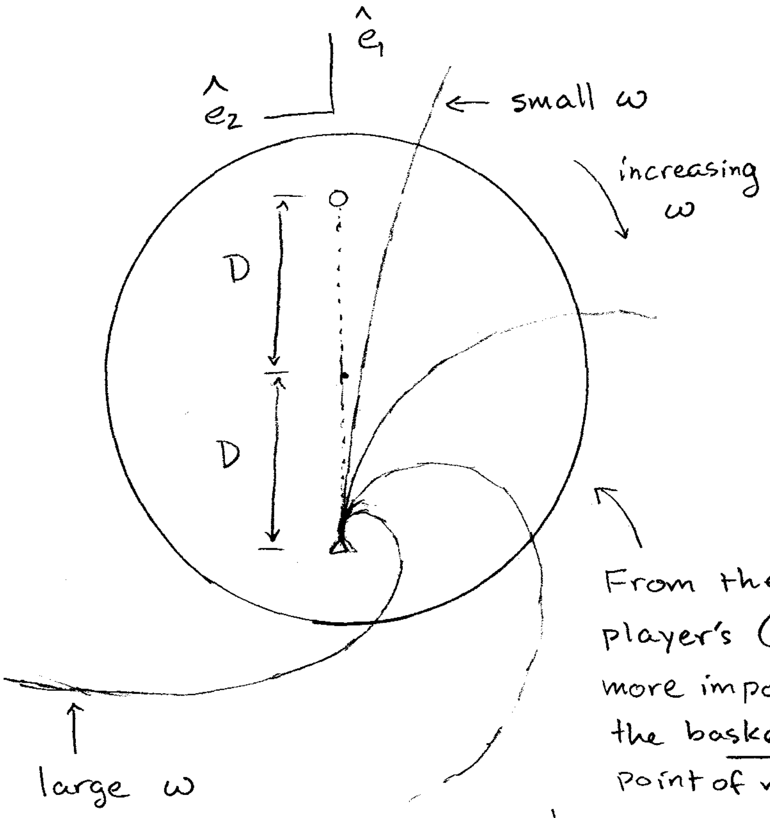
This is the acceleration with respect to an inertial frame.

In 2-D, while the basketball is in flight, we assume $\sum \vec{F} = 0$.

$$\Rightarrow \quad \ddot{x} - 2\omega\dot{y} - \omega^2 x = 0 \quad \text{AND} \quad \ddot{y} + 2\omega\dot{x} - \omega^2 y = 0$$

$$\Rightarrow \quad \boxed{\begin{aligned} \ddot{x} &= 2\omega\dot{y} + \omega^2 x \\ \ddot{y} &= -2\omega\dot{x} + \omega^2 y \end{aligned}}$$

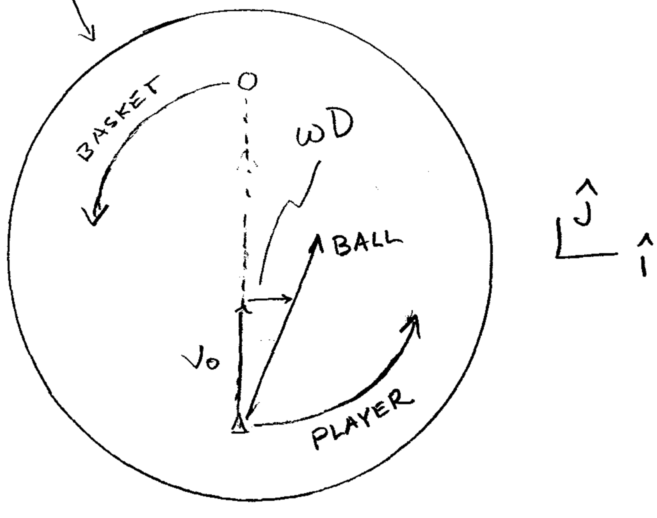
QUESTION: Why is it a good idea to describe things from the basket's point of view?



From the player's (and more importantly the basket's) point of view.

The player sees $x\hat{e}_1 + y\hat{e}_2$.

From our point of view.



We see

$$x(\cos\omega t\,\hat{j} - \sin\omega t\,\hat{i}) + y(-\cos\omega t\,\hat{i} - \sin\omega t\,\hat{j})$$

BUT BOTH ARE WATCHING THE SAME THING!

Centrifugal and Coriolis Forces

(inertial)

Let's say the player did not know she was spinning around. So, she assumes her reference frame B is fixed.

Then she would compute

$$\Sigma \vec{F} = m_Q \, {}^B\vec{a}_Q = m_Q(\ddot{x}\,\hat{e}_1 + \ddot{y}\,\hat{e}_2).$$

Then to explain what she saw (remember, the EOM's for the ball are $\ddot{x} = 2\omega\dot{y} + \omega^2 x$, $\ddot{y} = -2\omega\dot{x} + \omega^2 y$) she would have to assume that

$$\Sigma \vec{F} = m_Q\left[(2\omega\dot{y} + \omega^2 x)\,\hat{e}_1 + (-2\omega\dot{x} + \omega^2 y)\,\hat{e}_2\right]$$

$$= \underbrace{m_Q(2\omega\dot{y}\,\hat{e}_1 - 2\omega\dot{x}\,\hat{e}_2)}_{\text{CORIOLIS "FORCE"}} + \underbrace{m_Q(\omega^2 x\,\hat{e}_1 + \omega^2 y\,\hat{e}_2)}_{\text{CENTRIFUGAL "FORCE"}}$$

Are these forces real? No! Forces are caused by physical interaction. The coriolis and centrifugal "forces" are fake — they are non-physical, and only appear real to an observer who does not know that they are in a non-inertial reference frame. When asked what forces are acting on a system, never say coriolis or centrifugal. (Or at least, call them "fake forces".)

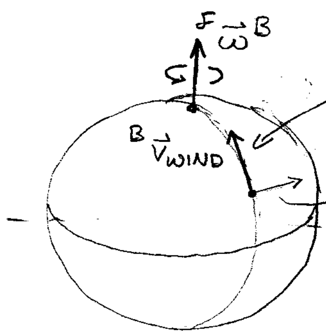
More generally, saying

$$\sum \vec{F} = m_Q \, {}^{\mathcal{F}}\vec{a}_Q$$

is equivalent to saying

$$\underbrace{\sum \vec{F}}_{\text{real forces}} \underbrace{- m_Q \, {}^{\mathcal{F}}\vec{a}_P - m_Q \, {}^{\mathcal{F}}\vec{\alpha}^B \times \vec{r}_{Q/P} - 2 m_Q \, {}^{\mathcal{F}}\vec{\omega}^B \times {}^B\vec{v}_{Q/P} - m_Q \, {}^{\mathcal{F}}\vec{\omega}^B \times \left({}^{\mathcal{F}}\vec{\omega}^B \times \vec{r}_{Q/P} \right)}_{\text{fake ``forces''}} = m_Q \, {}^B\vec{a}_Q$$

Nonetheless, fake forces can seem very real to someone sitting in frame B. For example, consider wind blowing across the surface of the earth. (Or think about playing basketball on a small rotating sphere.)



say wind is blowing north ...

$$\underbrace{-2 \, {}^{\mathcal{F}}\vec{\omega}^B \times {}^B\vec{v}_{WIND}}$$

the coriolis "force" makes the wind turn to the east, or at least explains why we observe the wind turning to the east.

Another (sometimes simpler) way to find velocity and acceleration of the basketball — just apply

$$\frac{{}^{\mathcal{F}}d\vec{u}}{dt} = \frac{{}^{B}d\vec{u}}{dt} + {}^{\mathcal{F}}\vec{\omega}^{B} \times \vec{u} \quad \text{twice.}$$

Method 2

$$\vec{r}_Q = x\hat{e}_1 + y\hat{e}_2 \quad \longleftarrow \text{(take point P as the origin)}$$

$$^{\mathcal{F}}\vec{v}_Q = \frac{{}^{\mathcal{F}}d\vec{r}_Q}{dt} = \frac{{}^{B}d\vec{r}_Q}{dt} + {}^{\mathcal{F}}\vec{\omega}^{B} \times {}^{B}\vec{r}_Q \quad \longleftarrow \text{FIRST TIME}$$

$$= (\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2) + (\omega\hat{k} \times (x\hat{e}_1 + y\hat{e}_2))$$

$$\Rightarrow \boxed{{}^{\mathcal{F}}\vec{v}_Q = (\dot{x} - \omega y)\hat{e}_1 + (\dot{y} + \omega x)\hat{e}_2}$$

$$^{\mathcal{F}}\vec{a}_Q = \frac{{}^{\mathcal{F}}d({}^{\mathcal{F}}\vec{v}_Q)}{dt} = \frac{{}^{B}d({}^{B}\vec{v}_Q)}{dt} + {}^{\mathcal{F}}\vec{\omega}^{B} \times {}^{\mathcal{F}}\vec{v}_Q$$

$$= \left[(\ddot{x} - \omega\dot{y})\hat{e}_1 + (\ddot{y} + \omega\dot{x})\hat{e}_2\right]$$

$$+ \left[\omega\hat{k} \times \left((\dot{x} - \omega y)\hat{e}_1 + (\dot{y} + \omega x)\hat{e}_2\right)\right]$$

$$= \left[(\ddot{x} - \omega\dot{y})\hat{e}_1 + (\ddot{y} + \omega\dot{x})\hat{e}_2\right] + \left[(-\omega\dot{y} - \omega^2 x)\hat{e}_1 + (\omega\dot{x} - \omega^2 y)\hat{e}_2\right]$$

$$\Rightarrow \boxed{{}^{\mathcal{F}}\vec{a}_Q = (\ddot{x} - 2\omega\dot{y} - \omega^2 x)\hat{e}_1 + (\ddot{y} + 2\omega\dot{x} - \omega^2 y)\hat{e}_2}$$

$\uparrow$

This computation was a bit simpler — we didn't need to keep track of so many terms (${}^{\mathcal{F}}\vec{\alpha}^{B}$, ${}^{B}\vec{v}_Q$, ${}^{B}\vec{a}_Q$, etc.).