AE 252 (Spring 2007)
Example: Hohmann Transfer

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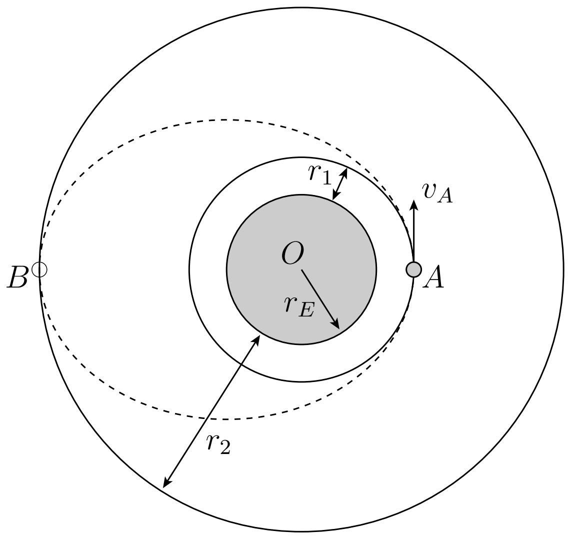

Problem. A satellite with total mass $m = 200$ kg is in a circular low-earth orbit at altitude $r_1 = 250$ km. It will transition to a circular medium-earth orbit at altitude $r_2 = 2500$ km. An intermediate elliptical orbit will be used to make the transfer. If the satellite has a rocket engine capable of supplying a constant thrust $F = 3000$ N, determine the length of the engine burns required at points A and B (the start and end of the elliptical transfer orbit). The radius of the earth is $r_E = 6378.1$ km.

Solution. The only force acting on the satellite is the gravitational force, with magnitude $F_{\text{grav}} = mgr_E^2/r^2$ and direction toward O (the center of the earth). In a uniform circular orbit, the satellite has acceleration with magnitude v^2/r and direction toward O. So we can use Newton's Law to find the speed of the satellite in any circular orbit:

$$\begin{aligned} \frac{mgr_E^2}{r^2} &= \frac{mv^2}{r} \\ \Rightarrow \quad v &= \sqrt{\frac{gr_E^2}{r}} = r_E\sqrt{\frac{g}{r}} \end{aligned}$$

From this expression, we know the speed in the low-earth orbit is

$$v_A = r_E\sqrt{\frac{g}{r_E + r_1}} = (6378.1 \times 10^3)\sqrt{\frac{9.81}{(6378.1 \times 10^3) + (250 \times 10^3)}} = 7.76 \times 10^3 \text{ m/s}$$

and the speed in the medium-earth orbit is

$$v_B = r_E\sqrt{\frac{g}{r_E + r_2}} = (6378.1 \times 10^3)\sqrt{\frac{9.81}{(6378.1 \times 10^3) + (2500 \times 10^3)}} = 6.70 \times 10^3 \text{ m/s}$$

Now what about the elliptical transfer orbit? Unlike for a circular orbit, the speed will *not* be constant. Let v_1 be the speed in the elliptical orbit just after the rocket is fired at A. Let v_2 be the speed in the elliptical orbit just before the rocket is fired at B. (We'll find out that $v_A \neq v_1$ and $v_B \neq v_2$, of course—this is why the rocket needs to be fired at all.)

In class we showed that both angular momentum and energy are conserved during orbital motion. We can combine these two facts to find v_1 and v_2. From conservation of angular momentum,

$$\begin{aligned} \vec{r}_1 \times m\vec{v}_1 &= \vec{r}_2 \times m\vec{v}_2 \\ \Rightarrow \quad (r_1 + r_E)v_1 &= (r_2 + r_E)v_2 \end{aligned}$$

From conservation of energy, where the potential for the gravitational force is $V = -mgr_E^2/r$,

$$\frac{1}{2}mv_1^2 - \frac{mgr_E^2}{r_1 + r_E} = \frac{1}{2}mv_2^2 - \frac{mgr_E^2}{r_2 + r_E}$$

These are two equations in two unknowns (v_1 and v_2). Combining them, we have

$$\begin{aligned} \frac{1}{2}mv_1^2 - \frac{mgr_E^2}{r_1 + r_E} &= \frac{1}{2}mv_1^2\frac{(r_1 + r_E)^2}{(r_2 + r_E)^2} - \frac{mgr_E^2}{r_2 + r_E} \\ \Rightarrow \quad v_1^2 - \frac{2gr_E^2}{r_1 + r_E} &= v_1^2\frac{(r_1 + r_E)^2}{(r_2 + r_E)^2} - \frac{2gr_E^2}{r_2 + r_E} \\ \Rightarrow \quad v_1^2\left(1 - \frac{(r_1 + r_E)^2}{(r_2 + r_E)^2}\right) &= 2gr_E^2\left(\frac{1}{r_1 + r_E} - \frac{1}{r_2 + r_E}\right) \\ \Rightarrow \quad v_1 &= \sqrt{\frac{2gr_E^2\left(\frac{1}{r_1+r_E} - \frac{1}{r_2+r_E}\right)}{\left(1 - \frac{(r_1+r_E)^2}{(r_2+r_E)^2}\right)}} \end{aligned}$$

Plugging in all the numbers, we find

$$\begin{aligned} v_1 &= \sqrt{\frac{2(9.81)(6378.1\times 10^3)^2\left(\frac{1}{(6378.1\times10^3)+(250\times10^3)}-\frac{1}{(6378.1\times10^3)+(2500\times10^3)}\right)}{\left(1-\frac{((6378.1\times10^3)+(250\times10^3))^2}{((6378.1\times10^3)+(2500\times10^3))^2}\right)}} \\ &= 8.30\times 10^3 \text{ m/s} \end{aligned}$$

And from conservation of angular momentum,

$$\begin{aligned} v_2 &= v_1\frac{r_1+r_E}{r_2+r_E} \\ &= (8.30\times 10^3)\frac{(6378.1\times 10^3)+(250\times 10^3)}{(6378.1\times 10^3)+(2500\times 10^3)} \\ &= 6.20\times 10^3 \text{ m/s} \end{aligned}$$

Now consider the first rocket burn, to accelerate the satellite from v_A to v_1 at point A. Using the principle of angular momentum, and assuming Δt is small enough to consider the position $\vec{r} = (r_1 + r_E)\hat{e}_r$ constant, we have

$$\begin{aligned} &\int_t^{t+\Delta t_1} ((r_1+r_E)\hat{e}_r \times F\hat{e}_\theta)dt = ((r_1+r_E)\hat{e}_r \times mv_1\hat{e}_\theta) - ((r_1+r_E)\hat{e}_r \times mv_A\hat{e}_\theta) \\ \Rightarrow\quad & (r_1+r_E)F\Delta t_1 = (r_1+r_E)m(v_1-v_A) \\ \Rightarrow\quad & \Delta t_1 = \frac{m(v_1-v_A)}{F} \end{aligned}$$

Plugging in the numbers, we find

$$\Delta t_1 = \frac{200\left((8.30\times 10^3)-(7.76\times 10^3)\right)}{3000} = 36.3 \text{ s}$$

Doing the same thing for the second rocket burn, we find

$$\Delta t_1 = \frac{m(v_B-v_2)}{F} = \frac{200\left((6.70\times 10^3)-(6.20\times 10^3)\right)}{3000} = 33.7 \text{ s}$$

Additional questions to consider. How long would the rocket burn have to be before you could no longer model it as an impulse? (In other words, when would this assumption start introducing considerable error?) How would you model an actual rocket burn with mass change? Is a Hohmann Transfer the most efficient way to get from one circular orbit to another?

Finally, in our solution we modeled the earth as stationary. The earth is not stationary (it's going very fast around the sun, and the sun is moving very fast itself). We also neglected the influence of the sun's gravity on the satellite, and the influence of the satellite's gravity on the earth. Are these assumptions reasonable? What happens if we relax them?