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Thermal-Viscoelastic Coupled Wave Motion

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The concept and mathematical modeling of thermo-elastic coupling was first formulated in 1837 [1] and detailed analyses and evaluations may be found in [2 – 5] among others. In the static elastic case such coupling effects only manifest themselves through the thermal expansions arising from the elastic constitutive relations, i.e. Coupling Term T_{4EL2} in (2). In the dynamic case the now time dependent displacements directly affect the temperature through their implicit appearances in the heat equations through Coupling Term T_{4EL} in (1). This additional term may be thought of as a virtual damping contribution, although the phase relations between temperature and displacements lead to either positive or negative "damping" influences in this totally conservative system.

In viscoelastic media, on the other hand, displacements are always time dependent due to the inherent material properties regardless of temperature-time dependencies and even in conjunction with static loads.. However, the additional Coupling Term T_{4VE}, though physically ever present, is seldom included in most analyses. The Term T_3 is predominantly the external heat flow supplied to or extracted from the body. However, it may also include dissipative viscoelastic contributions.

$$\text{conservation of thermal energy} \quad \Longrightarrow \quad \underbrace{k\,\frac{\partial^2 T(x_1,t)}{\partial x_1^2} - \rho\, c_v\,\frac{\partial T(x_1,t)}{\partial t} + \overbrace{q(x_1,t)}^{\substack{\text{external}\\ \text{heat flow } (\mathrm{T}_3)}}}_{\text{uncoupled rigid body heat conduction law [6]}}$$

$$- \left\{ \begin{array}{c} \underbrace{E_0\,\alpha\,T_0\,\frac{\partial^2 u_1^E(x_1,t)}{\partial x_1 \partial t}}_{\text{Coupling Term } \mathrm{T_{4EL}}\text{: elastic volume changes}} \\[2em] \underbrace{\alpha\,T_0\,\frac{\partial}{\partial t}\left(\int_{-\infty}^{t} E\left[t,t',T(x_1,t')\right]\;\frac{\partial^2 u_1(x_1,t')}{\partial x_1 \partial t'}\;dt' \right)}_{\text{Coupling Term } \mathrm{T_{4VE}}\text{: viscoelastic volume changes}} \end{array} \right\} = 0 \qquad (1)$$

and conservation of linear momentum $\Longrightarrow$

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$$\left\{\begin{array}{c} \underbrace{\rho\,\frac{\partial^2 u_1^E(x_1,t)}{\partial t^2}}_{\text{inertia force }(\mathrm{T_{1E2}})} \;+\; \underbrace{E_0\,\frac{\partial^2 u_1^E(x_1,t)}{\partial x_1^2}}_{\text{internal elastic stresses }(\mathrm{T_{2E2}})} \\[2em] \underbrace{\rho\,\frac{\partial^2 u_1(x_1,t)}{\partial t^2}}_{\text{inertia force }(\mathrm{T_{1VE2}})} \;+\; \underbrace{\int\limits_{-\infty}^{t} \frac{\partial}{\partial x_1}\left(E\left[t,t',T(x_1,t')\right]\;\frac{\partial^2 u_1(x_1,t')}{\partial x_1 \partial t'}\right)\,dt'}_{\text{internal viscoelastic stresses }(\mathrm{T_{2VE2}})} \end{array}\right\}$$

$$+\;\underbrace{f_1(x_1,t)}_{\substack{\text{body}\\ \text{force }(\mathrm{T_{3B2}})}}\;-\;\left\{\begin{array}{c} \underbrace{E_0\,\frac{\partial\left[\alpha T(x_1,t)\right]}{\partial x_1}}_{\text{Coupling Term }\mathrm{T_{4EL2}}\text{: stresses due to thermal expansions}} \\[2em] \underbrace{\int\limits_{-\infty}^{t} \frac{\partial}{\partial x_1}\left(E^T\left[t,t',T(x_1,t')\right]\;\frac{\partial\left[\alpha T(x_1,t')\right]}{\partial t'}\right)\,dt'}_{\text{Coupling Term }\mathrm{T_{4VE2}}\text{: stresses due to thermal expansions}} \end{array}\right\} \;=\; 0 \qquad (2)$$

Aside from the readily visible important fundamental differences in elastic and viscoelastic governing Eqs. (1) and (2), there remains the most significant matter of the temperature dependence of Young's and relaxation moduli. When one eliminates all relaxation/creep influences at elevated temperatures from Young's modulus experimental measurements, the remainder shows little variations of elastic moduli with temperature [7 – 8]. Viscoelastic relaxation moduli, on the other hand, show extreme sensitivity to temperature due to real material variations in viscosity coefficients of approximately one order of magnitude per 20°C. The most significant effect of this temperature dependence is to change the kernel functions in the hereditary integrals from $E(x,t-t')$ to $E(x,t,t')$ thus destroying the convenient properties of the convolution integrals.

While in other problems with TSM a transformation to a reduced time $\xi(x,t)$ defined by

$$\xi(x,t) \;=\; \int\limits_{0}^{t} a_T\left[x,t,^{*}T(x,t^{*})\right]\;dt^{*} \qquad (3)$$

yields a useful form of a convolution integral in the ξ-space such that

$$\int\limits_{-\infty}^{t} E\left[x,t,t',T(x,t')\right]\frac{\partial\epsilon_{11}(x,t')}{\partial t'}\,dt' \;=\; \int\limits_{-\infty}^{t} \widehat{E}(x,\xi-\xi')\frac{\partial\widehat{\epsilon}_{11}(x,\xi')}{\partial\xi'}\,d\xi' \qquad (4)$$

the presence of x_1 derivatives in Eqs. (1) and (2) fails to do so since

$$\frac{\partial^2 u_1(x,t')}{\partial x_1 \partial t'}\, dt' \;=\; \underbrace{\frac{\partial \xi'(x,t')}{\partial x_1}}_{=\; f(x_1,\xi')}\; \frac{\partial^2 \widehat{u}_1(x,\xi')}{\partial \xi'^2}\, d\xi' \tag{5}$$

where

$$f(x_1,\xi') \;=\; \int_0^{t'=F(x_1,\xi')} \frac{\partial a_T\left[x_1, t;^* T(x_1,t^*)\right]}{\partial x_1}\, dt^* \tag{6}$$

The viscoelastic constitutive relations and governing equations are linear if material properties are approximated as temperature independent and nonlinear when the more realistic temperature dependencies are included. The elastic formulations may be defined as linear because Young's modulus's (E_0) relatively weak temperature dependences generally can be ignored.

An elastic or viscoelastic solution to the governing relations (1) and (2) may be written in the form

$$u_1(x_1,t) \;=\; \sum_{m=1}^{M} \mathcal{U}_m(t)\; \sin\left(\frac{m\,\pi\,x_1}{L}\right) \qquad \text{and} \qquad T(x_1,t) \;=\; \sum_{m=1}^{M} \mathcal{T}_m(t)\; \sin\left(\frac{m\,\pi\,x_1}{L}\right) \tag{7}$$

where L is some predefined characteristic length. The application of Galerkin's method [9] eliminates the x_1 dependence in the governing relations and results in a viscoelastic system of time variable coefficient ordinary integral-differential equations. The symbolic Galerkin operations[2] can be carried out formally with such software programs as MATLAB™, MAPLE™, MATHEMATICA™, etc. Of course, the complete problem statement must include specific BCs and ICs.

The generalization to 3-D waves can be readily accomplished by appropriate extension of Eqs. (1) and (2). Viscoelastic effects of temperature-displacement coupling as well as the sensitivity of temperature to energy dissipation need to be extensively further investigated.

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